Scattering amplitudes and hidden symmetries in gauge theory and a duality to strings

Jan Plefka

Institut für Physik, Humboldt-Universität zu Berlin
und Institut für Theoretische Physik, ETH Zürich
Scattering amplitudes and hidden symmetries in supersymmetric gauge theory and a duality to strings

Plan:

1. Quantum field theory
2. Symmetries
3. Supersymmetric gauge field theory
4. String-gauge theory duality
5. Scattering amplitudes in gauge theories and on-shell methods
6. Hidden symmetries of scattering amplitudes
7. (Generalized unitarity)
Mathematical framework of particle physics: QFT

- **Quantum Field Theory**: Relativistic many particle quantum theory
- Describes scattering processes in accelerators

\[
e^− e^− = g^2 \cdot \text{Photon} + g^4 \cdot \text{Feynman diagram} + g^6 (\ldots) + \ldots
\]

Perturbative description: Series expansion in \( g \ll 1 \)  \( g \): Coupling constant

- **Feynman diagrams**: Describe particle propagation & interactions
- **Symmetries** play central role:
  - Determine possible particles & their interactions
  - Can severely constrain results for observables
- Exact analytic methods beyond perturbation theory are sparse
- Desirable to advance our fundamental understanding of quantum field theory
The Standard Model of Particle Physics

Three fundamental forces described by Gauge Field Theories [1955, 1971]

\[ \mathbf{SU(3)} \times \mathbf{SU(2)} \times \mathbf{U(1)} = \text{Gauge Field Theories} \]

Forces:

- Electromagnetism (photons)
- Weak Force (W & Z bosons)
- Strong Force (gluons) \( \cong \) Quantum Chromodynamics (QCD)

SU(\(N\)) Gauge Field Theory: Fields are \( N \times N \) matrices:

\[ A_{\mu}^{\mathbf{SU(2)}}(x) = \begin{pmatrix} Z & W^+ \\ W^- & -Z \end{pmatrix} \]

Spectrum:

<table>
<thead>
<tr>
<th>Leptons</th>
<th>Quarks</th>
<th>Vector bosons</th>
<th>Scalar</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e, \nu_e )</td>
<td>( u, d )</td>
<td>( A_{\mu} )</td>
<td>Higgs</td>
</tr>
<tr>
<td>( \mu, \nu_{\mu} )</td>
<td>( s, c )</td>
<td>( W^\pm, Z )</td>
<td></td>
</tr>
<tr>
<td>( \tau, \nu_{\tau} )</td>
<td>( t, b )</td>
<td>( A_{\mu}^{\alpha} )</td>
<td></td>
</tr>
</tbody>
</table>

Gravity is not contained!
Symmetries
Symmetries lie at the heart of our understanding of physics. They constrain or even determine physical theories and their observables.

- Mathematically symmetry transformations form a group
  \[ G_1 \circ G_2 = G_3 \quad \{1, G_i, G_i^{-1}\} \in \text{group} \]
- Continuous transf.: Lie group \( G(\phi) = e^{i \phi^a \hat{J}_a} \) \( \hat{J}_a \) : Generator \( \phi^a \in \mathbb{R} \)
- Group property entails commutation relations
  \[ [\hat{J}_a, \hat{J}_b] = i f_{ab}^c \hat{J}_c \quad \text{Lie algebra} \quad a, b, c = 1, \ldots, \text{dim}(g) \]
- Symmetries can be obvious or hidden
- Example for obvious symmetries: Rotations and translations
  \[ R(\vec{\phi}) = e^{i \vec{\phi} \cdot \vec{L}} \quad \vec{L} : \text{Angular momentum} \]
  \[ T(\vec{\alpha}) = e^{i \vec{\alpha} \cdot \vec{P}} \quad \vec{P} : \text{Momentum} \]
  \[ [L_i, L_j] = i \hbar \epsilon_{ijk} L_k \quad [P_i, P_j] = 0 \quad [L_i, P_k] = i \hbar \epsilon_{ijk} P_k \]
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Example of a hidden symmetry: The Hydrogen atom

- Hamiltonian: \( H = \frac{\vec{p}^2}{2m} - \frac{k}{r} \)

- **Obvious** rotational symmetry: \([H, L_i] = 0 \quad \Rightarrow \quad H |n, l, m\rangle = E_{n,l} |n, l, m\rangle\)

- **Hidden** symmetry in H-atom: Pauli-Lenz vector

  \[ \vec{A} = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - m k \frac{\vec{r}}{r} \]

- Conserved quantity: \([H, A_i] = 0\)

- Algebra:

  \[ [A_i, A_j] = -i \frac{2\hbar}{m} H L_k , \quad [L_i, A_j] = i \hbar \epsilon_{ijk} A_k , \quad [L_i, L_j] = i \hbar \epsilon_{ijk} L_k \]

- Closes on eigenspace \( \mathcal{H}_E \) of fixed energy eigenvalue \( E \).

- Operator algebra determines spectrum (\( \cong \) representation theory of \( SU(2) \))

  \[ E_n = -\frac{mk^2}{2\hbar^2} \frac{1}{n^2} \quad (\text{degeneracy } n^2) \]
Fundamental symmetry of QFT: Poincaré group

Poincare symmetry: Lorentz transformations + translations in space & time

\[ M_{\mu\nu}, P_\mu \]

\[ \vec{L} \]
rotations

\[ \vec{K} \]
boosts

\[ \begin{align*}
L_i &= \frac{1}{2} \epsilon_{ijk} M_{jk} \\
K_i &= M_{0i} \\
\mu, \nu, \ldots &= \{0, i\}
\end{align*} \]

- Representations of the 4d Poincaré group \( \hat{\mathcal{G}} \) possible particles in nature

<table>
<thead>
<tr>
<th>spin</th>
<th>field</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>scalar ( \phi(x) )</td>
<td>Higgs</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>left handed spinor ( \chi_\alpha(x) )</td>
<td>leptons, quarks</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>right handed spinor ( \bar{\psi}_\dot{\alpha}(x) )</td>
<td>leptons, quarks</td>
</tr>
<tr>
<td>1</td>
<td>vector ( A_\mu(x) )</td>
<td>photon, gauge bosons</td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>( \psi^\alpha_\mu(x) )</td>
<td>gravitino</td>
</tr>
<tr>
<td>2</td>
<td>( h_{\mu\nu}(x) )</td>
<td>graviton</td>
</tr>
</tbody>
</table>

- Massless fields/particles classified by helicity \( h = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|} \) with \( h = \pm s \)
Extension I: Conformal symmetry

- Relativistic QFTs without intrinsic mass scale (≈ massless or at very high energies) have an enlarged space-time symmetry: **Conformal symmetry**

- New transformations: Dilatations and inversions

<table>
<thead>
<tr>
<th>Dilatation transf.:</th>
<th>$D: \quad x^\mu \rightarrow \kappa x^\mu \quad \kappa \in \mathbb{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Special conformal transf.:</td>
<td>$K^\mu = I \circ P^\mu \circ I$ with $I: \text{Inversion} \quad x^\mu \rightarrow \frac{x^\mu}{x^2}$</td>
</tr>
</tbody>
</table>

Angle preserving transformations

- Conformal group is $SO(2, 4)$ with algebra:

  $[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})$,  
  $[D, P_\mu] = iP_\mu$,  
  $[D, K_\mu] = -iK_\mu$,  
  $[K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu)$  
  & Poincaré algebra

- Prominent examples:

  - Maxwell’s theory  \quad $\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$,  
    \quad $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
  
  - $\lambda \phi^4$ theory  \quad $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \lambda \phi^4$

  - Standard model  \quad $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} D \psi + \psi_i Y_{ij} \psi_j \phi$

  up to Higgs mass term

  \quad $+ |D_\mu \phi|^2 - \lambda |\phi|^4 - m^2 |\phi|^2$
Supersymmetry is a unique extension of space-time symmetries [1971,1974]

"Square root" of the momentum: \( \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2 (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \)

- Graded Lie algebra: Generators \( Q_\alpha \) & \( \bar{Q}_{\dot{\alpha}} \) are fermionic. Obey Super-Poincaré algebra with generators \( \{M_\mu, P_\mu; Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \)
- Relates bosons and fermions:
  \[
  \bar{Q}_{\dot{\alpha}} |\text{spin} = s\rangle = |\text{spin} = s + 1/2\rangle
  \]
- SUSY: \( \text{Boson} \leftrightarrow \text{Fermion} \)
  \( \text{Gluon} \leftrightarrow \text{Gluino} \)

Superpartners are degenerate in all quantum numbers (mass, charge, ...)

- **Extended supersymmetry**: Can have more than one set of supercharges
  \[ Q^A_\alpha \text{ & } \bar{Q}_{\dot{\alpha}}^A \text{ with } A = 1, \ldots, N: \]
  \[
  B_1 \leftrightarrow F_1 \\
  B_2 \leftrightarrow F_2
  \]
  \( \text{Gluon} \leftrightarrow N \text{ Gluinos} \)

- Maximal SUSY: \( \mathcal{N} = 4 \)  
  spin-range \( \{-1, -1/2, 0, 1/2, 1\} \)
What is the simplest gauge theory?

Maximally supersymmetric Yang-Mills theory

- Most (super)symmetric theory possible (without gravity)
- Uniquely specified by local internal symmetry group - e.g. number of colors $N_c$ for $SU(N_c)$
- Exactly scale-invariant field theory for any coupling (Green function scales with distance)
- Weak/strong coupling duality (AdS/CFT correspondence, gauge/string duality)

Particle content:
- massless spin-1 gluon (= the same in QCD)
- 4 massless spin-1/2 gluinos (= cousins of the quarks)
- 6 massless spin-0 scalars

Interaction between particles:
- All proportional to same dimensionless coupling $g_{YM}$ and related to each other by supersymmetry

sometimes called "harmonic oscillator/hydrogen atom of QFT"
Gauge Field Theory (or Yang-Mills-Theory)

- Builds upon internal (non-space-time) symmetry
- **SU(N)** Gauge theory: \([1954]\)
  
  Generalization of Maxwell’s theory of electromagnetism: \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\)

  Vector potential now \(N \times N\) hermitian matrix: \((A_\mu)_{ab}(x)\) \(a,b=1,...,N\)

- **Local** gauge symmetry:
  \[A_\mu(x) \rightarrow U A_\mu U^\dagger + i g U \partial_\mu U^\dagger\]
  \[\partial_\mu = \frac{\partial}{\partial x^\mu}\]

  with \(U \in SU(N)\), i.e. unitary \(N \times N\) matrix, \(U U^\dagger = 1\)

- Invariant action
  \[S_{YM} = \frac{1}{4} \int d^4 x \text{Tr}(F_{\mu \nu} F^{\mu \nu})\]
  \[F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu]\]

  \(g\): Coupling constant.

- \(N = 1\): Maxwell theory!
\( \mathcal{N} = 4 \) super Yang-Mills theory (SYM)

Can we have everything?

- Poincaré symmetry \( \rightarrow \) relativistic QFT
- Conformal symmetry \( \rightarrow \) scale-invariant QFT
- Maximal supersymmetry \( (\mathcal{N} = 4) \)
- \( SU(N) \) local gauge symmetry (with \( N \to \infty \))

\[
\begin{align*}
A_\mu & \quad \text{1 Gluon} \quad \text{spin=1} \quad \cdots \quad (= \text{same as in QCD}) \\
\psi_A^\alpha & \quad \text{4 Gluinos} \quad \text{spin=1/2} \quad \cdots \quad (= \text{cousin of the quarks}) \\
\phi_I & \quad \text{6 Scalars} \quad \text{spin=0} \quad \cdots \cdots \\
\end{align*}
\]

\[
\mathcal{L}_{\text{SYM}} = \frac{N}{\lambda} \text{Tr} \left[ \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \Phi_I)^2 - \frac{1}{4} [\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \bar{\psi} \slashed{D} \psi + \psi A \psi^B \Phi_{AB} + h.c. \right]
\]

Interactions:

All fields \( N \times N \) matrices. In \( N \to \infty \) (planar) limit: One parameter \( \lambda = g^2 N \)
The simplest quantum field theory

$\mathcal{N} = 4$ SYM has remarkably rich properties:

- Uniquely determined by $g_{YM}$ & $N$, exactly scale invariant at any coupling, no UV divergences $\Rightarrow g_{YM} = \text{const}$ [1980’s]
- **Dual** to string theory $\rightarrow$ **AdS/CFT correspondence.** [1997]
  - Strong coupling limit ($\lambda = g^2 N^2 \rightarrow \infty$): Classical string on $AdS_5 \times S^5$.
- Appears to be **integrable** in $N \rightarrow \infty$ limit: [since 2003]
  - Exact results for two-point correlation functions $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = (x - y)^{2\Delta + \gamma(\lambda)}$
  - Hidden symmetries beyond super-conformal group: Yangian algebra
  - Deep mathematical understanding of scattering amplitudes

$\Rightarrow$ **An ideal theoretical laboratory** to study gauge theories (and string theory)!
$\Rightarrow$ Could be the first exactly solvable interacting 4d QFT.
$\Rightarrow$ **Non-physical!** But possible starting point for novel perturbative approach.
$\Rightarrow$ Already now application to massless QCD exist.
The world as a hologram
String theory in a nut-shell

- **Idea:** Replace particle by extended 1d object: **string**

  ![String](image)

- Quantum mechanics of a relativistic string in **flat space-time** $\mathbb{R}^{1,d-1}$:

  - Graviton
  - Gauge boson
  - Matter particle

- Oscillation spectrum $\hat{\omega}$ corresponds to spectrum of “elementary particles”

- **Strings** must propagate in $d=9+1$. Theory depends on **background geometry**

- Consistent theory of quantum gravity

- Unification of matter and force particles as excitation of one entity: the fundamental string
Spectrum of string excitations in flat (Minkowski) space-time

String theory $\cong$ QFT with infinite number of particle species

- Graviton
- Gravitino
- Echteilchen
- Spinor
- Skalar

Spin

Low energy limit: Recovers known interacting particle field theories

This is the well understood situation in flat $\mathbb{R}^{1,9}$

Now: Take curved space-time geometry with boundary!
Quantum gravity in a box

- Space-time with negative curvature: **anti-de-Sitter space** \((AdS_d)\)
  
  [Willem de Sitter, 1872-1934]

- \(AdS_5\) is (4+1)-dimensional space-time with **boundary** of geometry \(\mathbb{R}^{1,3}\)

- String theory well defined on \(AdS_5 \times M_5\), e.g. \(M_5 = S^5\) (5d-sphere).

- **Quantization** of strings on \(AdS_5 \times S_5\) unsolved!

- Isometry group of \(AdS_5\) \(\cong\) conformal group in 4d
**Holographic duality:** Strings in $AdS_5 \times S_5$ are dual to $\mathcal{N} = 4$ SYM

- **Claim:** Two alternative mathematical descriptions of one physical object!
The dual model: Superstring in $AdS_5 \times S^5$

\[
S_{\text{string}} = \sqrt{\lambda} \int d\tau d\sigma \left[ G^{(AdS_5)}_{mn} \partial_a X^m \partial^a X^n + G^{(S^5)}_{mn} \partial_a Y^m \partial^a Y^n + \text{fermions} \right]
\]

- $ds^2_{AdS} = R^2 \frac{dx_3^2 + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, semiclassical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $\lambda \ll 1$: perturbative SYM \hspace{1cm} vs. \hspace{1cm} $\lambda \gg 1$: semiclassical ST
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- **Isometries:** $\mathfrak{so}(2, 4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2, 2|4)$
- Quantization unsolved!
Gauge Theory Observables

- **Local operators:** \( \mathcal{O}_n(x) = \text{Tr}[\mathcal{W}_1 \mathcal{W}_2 \ldots \mathcal{W}_n] \) with \( \mathcal{W}_i \in \{ \mathcal{D}^k \Phi, \mathcal{D}^k \Psi, \mathcal{D}^k F \} \)

2 point fct: \( \langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \rangle = \frac{\delta_{ab}}{(x_1 - x_2)^2 \Delta_a(\lambda)} \Delta_a(\lambda) \) Scaling Dims

- **Wilson loops:**

\[ \mathcal{W}_C = \langle \text{Tr} \ P \exp i \int_C ds (\dot{x}^\mu A_\mu + i|\dot{x}| \theta^I \Phi_I) \rangle \]

- **Scattering amplitudes:**

\[ \mathcal{A}_n(p_i, h_i, a_i; \lambda) = \begin{cases} \text{UV-finite} \\
\text{IR-divergent} \end{cases} \]

helicities: \( h_i \in \{0, \pm \frac{1}{2}, \pm 1\} \)
\[ \Delta_a(\lambda) \text{ spectrum of scaling dimensions} \]

\[ \mathcal{A}_n(p_i, h_i, a_i; \lambda) \]

\[ E(\lambda) \text{ string excitation spectrum (solved?)} \]

Wilson loop \( \mathcal{W}_C \)

T-dual

\[ \Sigma \]

light-like boundary

\[ z=0 \]
Scattering amplitudes
Scattering amplitudes

\[ A_n(\{p_i, h_i\}) = \text{probability amplitude for scattering process} \]

Central quantum field theory prediction for collider experiments

Computed via Feynman diagrams:

**Propagator**

\[ \frac{\delta^{ab} \eta_{\mu \nu}}{k^2 + i\epsilon} \quad \text{(gluons)} \]

**Vertices**

\[ g f^{abc} \left[ (q - r)_\mu \eta_{\nu \rho} + (r - p)_\nu \eta_{\rho \mu} + (p - q)_\rho \eta_{\mu \nu} \right] \]

\[ = -ig^2 \left[ f^{abe} f^{cde} (\eta_{\mu \rho} \eta_{\nu \sigma} - \eta_{\mu \sigma} \eta_{\nu \rho}) + f^{ace} f^{dbe} (\eta_{\mu \sigma} \eta_{\rho \nu} - \eta_{\mu \nu} \eta_{\rho \sigma}) + f^{ade} f^{bce} (\eta_{\mu \nu} \eta_{\sigma \rho} - \eta_{\mu \rho} \eta_{\sigma \nu}) \right] \]
Feynman diagramatic expansion

Task:

a) Draw all Feynman diagrams contributing to a given process
b) Integrate over all internal (off-shell) momenta $\int d^{4-2\epsilon}l$ imposing momentum conservation $\delta^{(4)}(\sum_i p_i)$ at each vertex
c) $A_n = \sum$ all diagrams

Can rapidly get out of hand: (even at tree-level)

<table>
<thead>
<tr>
<th>number of external gluons</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of diagrams</td>
<td>4</td>
<td>25</td>
<td>220</td>
<td>2485</td>
<td>34300</td>
<td>559405</td>
<td>10525900</td>
</tr>
</tbody>
</table>
Result of a brute force calculation (actually only a small part of it):

\[ k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5 \]
Simplicity of the result

When expressed in right variables the result is remarkably simple: 

\[ A_5(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0 \]

\[ A_5(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \]

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(all others from cyclicity and parity)

Spinor helicity: 

\[ p^\mu \rightarrow p^{\alpha \dot{\alpha}} = \bar{\sigma}_\mu^{\alpha \dot{\alpha}} p^\mu = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \]

(makes \( p^\mu p_\mu = 0 \) manifest)

\[ \lambda^\alpha = \frac{1}{\sqrt{p^0 + p^3}} \left( \begin{array}{c} p^0 + p^3 \\ p^1 + ip^2 \end{array} \right), \quad \tilde{\lambda}^{\dot{\alpha}} = (\lambda^\alpha)^\dagger, \quad \langle ij \rangle = \epsilon_{\alpha \beta} \lambda^{\alpha} \lambda^{\beta} \]

What is the reason for this simplicity?

- Hidden symmetries  \((\rightarrow \text{hidden super-conformal invariance & more})\)
- Analytic structure of the amplitude  \((\rightarrow \text{factorization, soft & collinear limits})\)
Simplicity of the result

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\[ \lambda^\alpha = \frac{1}{\sqrt{p^0 + p^3}} \left( \begin{array}{c} p^0 + p^3 \\ p^1 + i p^2 \end{array} \right), \quad \tilde{\lambda}^{\dot{\alpha}} = (\lambda^\alpha)^\dagger, \quad \langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta \]

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- Analytic structure of the amplitude  \((\rightarrow\) factorization, soft & collinear limits\))
Basic problem of Feynman diagramatic approach

In Feynman graph techniques one sums and integrates over non-physical terms:

\[ \int \frac{d^3 p \, dE}{(2\pi)^4} \]

Internal states are off-shell, violate mass-shell condition

Similarly individual diagrams are gauge variant, but final result is gauge invariant!

**On-shell approaches:**

Since 2005 tremendous progress in our understanding of scattering amplitudes based on on-shell formulations:

- On-shell recursion relations \( \checkmark \)
- Hidden symmetries \( \checkmark \)
- Generalized unitarity \( \checkmark \)
- Twistors & the Grassmannian & Integrability \( \times \)

The \( \mathcal{N} = 4 \) SYM theory has been instrumental in this progress!
Britto-Cachazo-Feng-Witten (BCFW) recursion

- **Idea**: Complexify momenta but stay on-shell \( z \in \mathbb{C} \)
  \[
p_1 \rightarrow \hat{p}_1(z) = \lambda_1 (\tilde{\lambda}_1 - z \tilde{\lambda}_n) \quad p_n \rightarrow \hat{p}_n(z) = (\lambda_n + z \lambda_1) \tilde{\lambda}_n
  \]
  Obeys \( \hat{p}_i(z)^2 = 0 \) and \( \hat{p}_1(z) + p_2 + \ldots + p_{n-1} + \hat{p}_n(z) = 0 \).

  - Deformation \( \mathcal{A}_n \rightarrow \mathcal{A}_n(z) \quad \mathcal{A}_n(z = 0) = \int dz \mathcal{A}_n(z)/z \)
  - Cauchy’s theorem yields recursive relation for on-shell amplitudes
  \[
  \mathcal{A}_n = \sum_i \mathcal{A}_{i+1} \frac{1}{\hat{P}_i^2} \mathcal{A}_{n-i+1}
  \]

  \[
  \sum \quad \begin{array}{c}
  A_L \rightarrow \hat{P}_i \rightarrow A_R
  \end{array}
  \quad \hat{P}_i^2 = 0
  \]

  - “Atoms” are the 3-point amplitudes: \( A_3(i^-, j^-) = \frac{\langle ij \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \)

  - Example:

\[
\begin{align*}
\mathcal{A}_{6:3} &= \mathcal{A}_{5:2} + 2 \mathcal{A}_{4:2} + \mathcal{A}_{3:1} \mathcal{A}_{5:3} \\
&= \mathcal{A}_{5:2} + 2 \mathcal{A}_{4:2} + \mathcal{A}_{3:1} \mathcal{A}_{5:3}
\end{align*}
\]
\( \mathcal{N} = 4 \) SYM: Superamplitudes and Super-BCFW recursion

- Consider super momentum-space using 4 anti-commuting coordinates \( \eta^A \):

  \[
  q^\alpha A = \lambda^\alpha \eta^A
  \]

  \[
  p^{\alpha \dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}
  \]

- Define superamplitudes \( \mathbb{A}_n \) in this formal space:

  \[
  \mathbb{A}_n = \frac{\delta^{(4)}(\sum_i p_i) \delta^{(8)}(\sum_i q_i)}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})
  \]

Superamplitudes package all gluon-gluino-scalar amplitudes.

- Super-BCFW recursion exists:

  \[
  \mathbb{A}_n(1, \ldots, n) = \sum_{i=3}^{n-1} \int d^4 \eta \hat{P}_i \mathbb{A}^L_i(1, \ldots, -\hat{P}_i) \frac{1}{P_i^2} \mathbb{A}^R_{n-i+2}(\hat{P}_i, \ldots, \hat{n})
  \]

- May be solved analytically!

⇒ All tree-amplitudes in \( \mathcal{N} = 4 \) SYM known in analytic form.
Application to massless QCD

Use gluon-gluino amplitudes from $\mathcal{N} = 4$ SYM to construct compact analytic formulae for all $n$-point tree-level gluon-quark $(g^{n-2l}(q\bar{q})^l)$ amplitudes with $l \leq 4$:

\[ \text{[Dixon, Henn, JP, Schuster]} \]

Needs to suppress intermediate production of scalars

\[
\begin{align*}
A & \pm B + A - B \\
A & \pm B - B + A -
\end{align*}
\]

Leads to numerically fast and stable results

\[ \text{[Badger, Biedermann, Hackl, JP, Schuster, Uwer]} \]

\( \Rightarrow \) Mathematica package GGT available \[ \text{[Dixon, Henn, JP, Schuster]} \]

\( \Rightarrow \) Formulae are being used for cross section computations of LHC processes today!

\[ \text{[BlackHat collaboration]} \]
Symmetries of scattering amplitudes

- Superconformal symmetry of $\mathcal{N} = 4$ SYM constrains superamplitudes

$$\mathbb{A}_{\text{tree}}^n = \frac{\delta^{(4)}(\sum_i p_i) \delta^{(8)}(\sum_i q_i)}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \mathcal{P}_n(\lambda_i, \tilde{\lambda}_i, \eta_i)$$

- Obvious symmetries:

$$p^{\alpha \dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \quad q^{\alpha A} = \sum_{i=1}^{n} \lambda_i^\alpha \eta_i^A \quad \Rightarrow \quad p^{\alpha \dot{\alpha}} \mathbb{A}_{\text{tree}}^n = 0 = q^{\alpha A} \mathbb{A}_{\text{tree}}^n$$

Explain vanishing of $A_n(1^\pm, 2^+, \ldots, n^+)$

- Less obvious symmetries

$$k_{\alpha \dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \quad s_{\alpha A} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \eta_i^A} \quad \Rightarrow \quad k_{\alpha \dot{\alpha}} \mathbb{A}_{\text{tree}}^n = 0 = s_{\alpha A} \mathbb{A}_{\text{tree}}^n$$

Explain form of $A_n(1^-, 2^-, 3^+ \ldots, n^+)$

- Super-conformal invariance of tree-amplitudes ($32 + 32$ generators):

$$J^a \mathbb{A}_{\text{tree}}^n = 0 \quad \text{with} \quad J^a \in \{p, k, \bar{m}, m, d, r, q, \bar{q}, s, \bar{s}, c_i\}$$
Infinite dimensional hidden symmetry

Tree superamplitudes are invariant under additional hidden symmetry (as in H-atom) [Drummond,Henn,Korchemsky,Sokatchev][Drummond,Henn,JP]

- **Mathematical structure:** *Yangian algebra* $Y[psu(2,2|4)]$ [Drinfeld]

\[ J^a = \sum_{i=1}^{n} J^a_i \quad \text{(level 0)} \quad J^a_{(1)} = f^a_{\; bc} \sum_{i<j}^{n} J^b_i J^c_j \quad \text{(level 1)} \]

An $\infty$-dim non-local symmetry algebra $J^a_{(n)} \; n = 0, 1, 2, \ldots$

\[
\begin{align*}
[J^a, J^b] &= if^{ab}_{\; c} J^c \\
[J^a, J^b_{(1)}] &= if^{ab}_{\; c} J^c_{(1)} \\
[J^a_{(1)}, J^b_{(1)}] &= if^{ab}_{\; c} J^c_{(2)} + g_{ab}(J^a_{(1)}, J^a_{(1)}) \\
\end{align*}
\]

\[ J^a_{(n)} A_{\text{tree}}^n = 0 \quad \forall n \] [Drummond,Henn,JP]

- **Signature of integrable field theory.** Explains simplicity of $A_{\text{tree}}^n$
  \[ \iff \text{Determines form of } A_{\text{tree}}^n \] [Bargheer,Beisert,McLoughlin,Loebbert,Galleas]

- **AdS/CFT:** T-duality of dual string theory. [Alday,Maldacena][Beisert,Ricci,Tseytlin][Berkovits,Maldacena]
Recent developments

- **Tree level** scattering amplitudes \( \hat{=} \) Sum of Yangian invariants

\[
|\Psi\rangle_{n,p} = \mathcal{B}_{i_1 j_1}(u_1) \ldots \mathcal{B}_{i_p j_p}(u_p)|0\rangle
\]

Based on methods of “quantum inverse scattering method” \( \Rightarrow \) Towards an algebraic S-Matrix [Chicherin, Derkachov, Kirschner] [Kanning, Lukowski, Staudacher][Broedel, de Leuw, Rosso]

- **Loop level** scattering amplitudes:

  - IR divergencies break conformal symmetry in a controlled way: **Conformal anomaly** [Drummond, Henn, Korchemsky, Sokatchev]

  - Deformed Yangian symmetry a 1-loop level [Beisert, Henn, McLoughlin, JP]

  - All loop integrands are Yangian invariant and constructible via loop level BCFW recursion [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]

  - And more . . .
Generalized unitarity

- On-shell methods also constructive at 1-loop (NLO-order) \([\text{Bern,Dixon,Dunbar,Kosower}]\)
- General 1-loop amplitude may be decomposed in basis integrals
  \[ [\text{Passarino,Veltman}][\text{Ossola,Papadopoulos,Pittau}][\text{Giele,Kunszt,Melnikov}] \]

![Diagrams]

- In \( \mathcal{N} = 4 \) SYM: Only box integrals occur due to hidden symmetry.
  \[ A_{n}^{1\text{-loop}} = \sum_{i} c_{i} \text{Box}_{i} \]

- Find \( c_{i} \) by putting internal propagators on-shell
  \[ c_{i} = \frac{1}{2} \sum_{l_{\pm}} A_{1}^{\text{tree}}(l_{\pm}) A_{2}^{\text{tree}}(l_{\pm}) A_{3}^{\text{tree}}(l_{\pm}) A_{4}^{\text{tree}}(l_{\pm}) \]
State of the art

Known MHV amplitudes: $A_n(1^-, 2^-, 3^+, \ldots, n^+)$ in $\mathcal{N} = 4$ SYM

AdS/CFT
[Alday,Maldacena]

Bern-Dixon-Smirnov ansatz & dual conformal symmetry
[Anastasiou,Bern,Dixon,Kosower][Drummond,Henn,Sokatchev,Korchemsky]

bootstrap [Drummond,Dixon,Duhr, Pennington,Hippel]
& integrability [Basso-Sever-Vieira]

integrands (loop level recursion)
[Arkani-Hamed et al]

unitarity

BCFW recursion
Field combines a multitude of areas in theoretical and mathematical physics:

- Phenomenology of elementary particles
- Integrable systems
- Fundamental aspects of quantum field theory
- Mathematics: Algebraic geometry & number theory
- String Theory

⇒ Intellectually rich and fascinating research area with “real physics” applications!
Thank you for your attention

Literature:

Bern, Dixon, Kosower, Scientific American 2012
Henn & Plefka, „Scattering Amplitudes in Gauge Theories“
LNP 883, Springer