5. Schätzung und Anpassung von Paramteren

5.1 Problemstellung
5.2 Beispiele einer Stichprobenfunktion
5.3 Die χ^2-Verteilung
5.4 Methode der kleinsten Qudarate

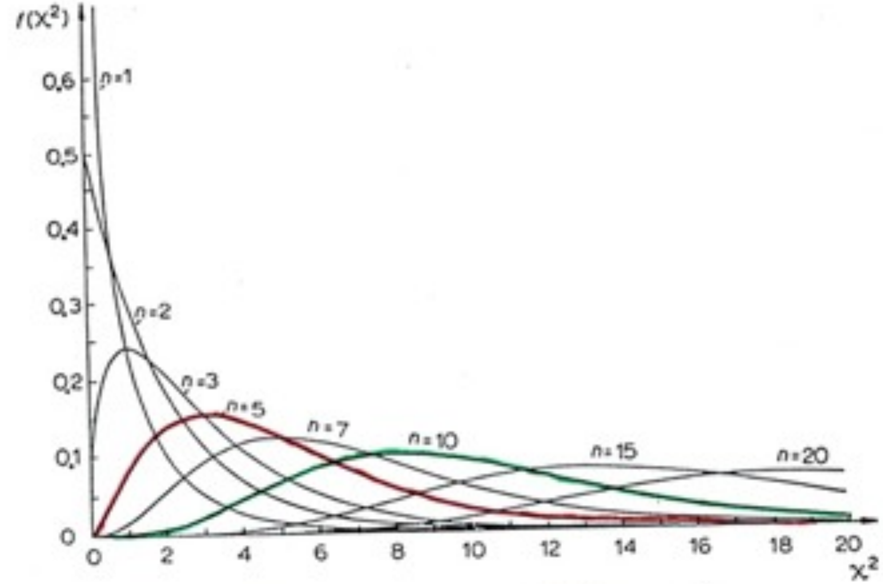


Bild 6.2 Wahrscheinlichkeitsdichte von χ^2.

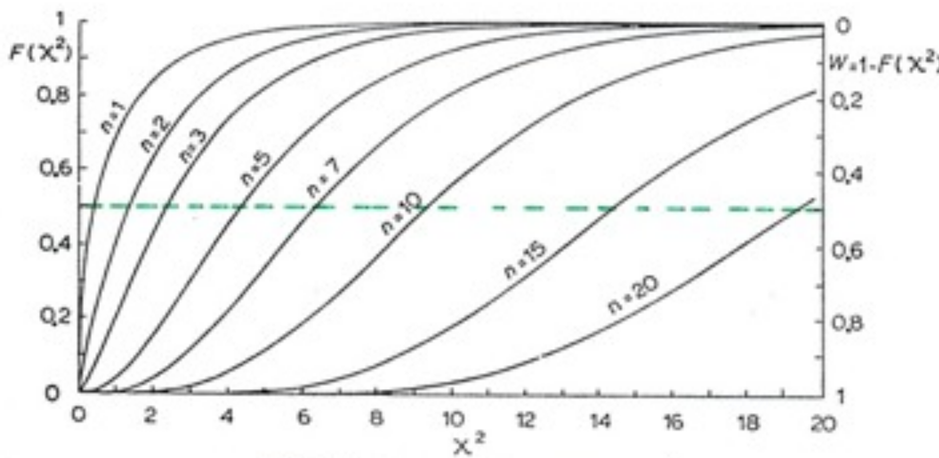
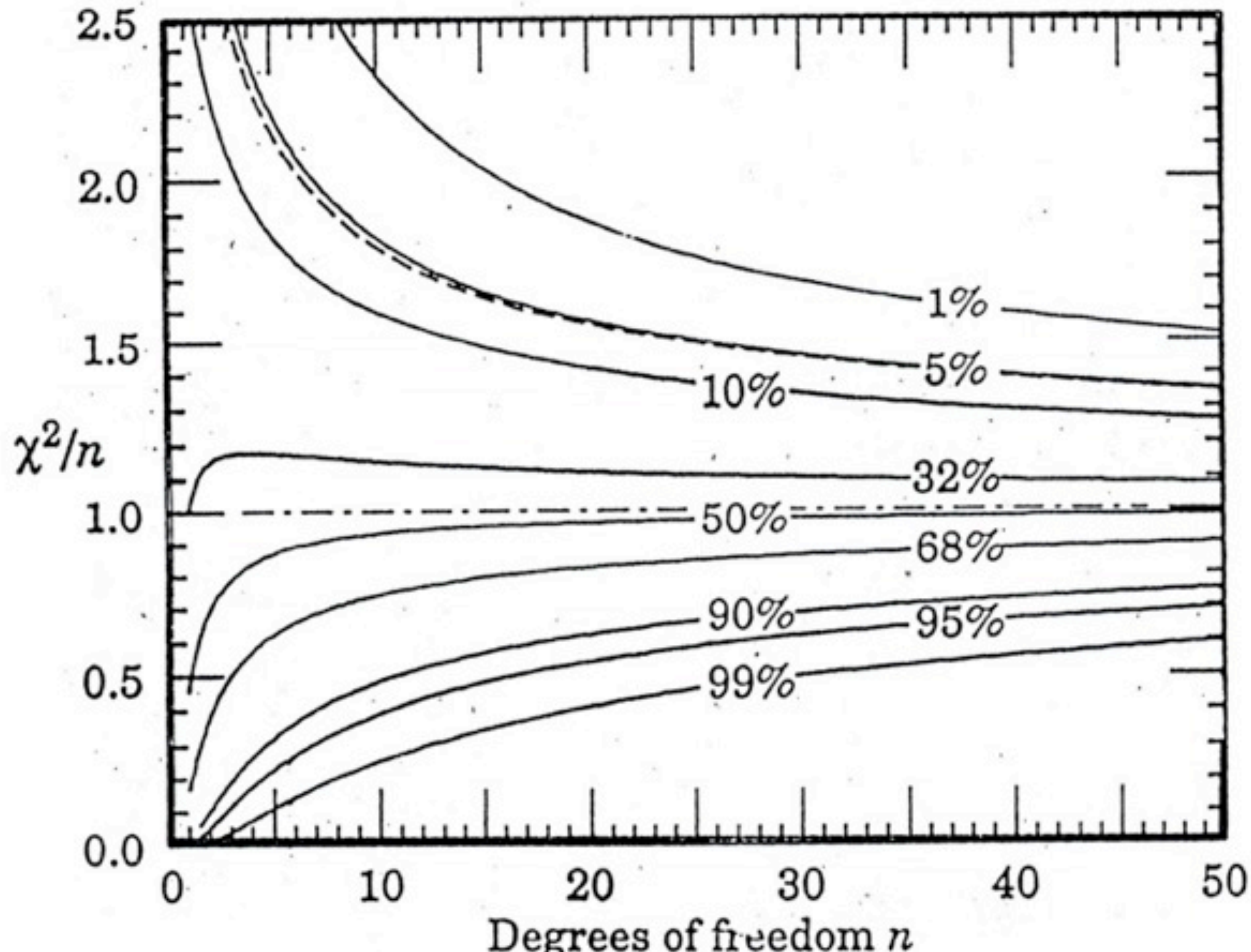


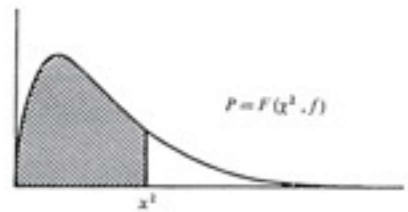
Bild 6.3 Verteilungsfunktion von χ^2.


2.5
2.0
1.5
1.0
0.5
0.0
χ²/n
0
10
20
30
40
50
Degrees of freedom n
1%
5%
10%
32%
50%
68%
90%
95%
99%

Tafel F – 4
χ^2–Verteilung.

Tabelliert sind die Werte P, definiert durch

$$P = F(\chi^2, f) = \frac{1}{\Gamma(\frac{1}{2}f)\,2^{\frac{1}{2}f}} \int_0^{\chi^2} u^{\frac{1}{2}f-1} e^{-\frac{1}{2}u}\,du$$

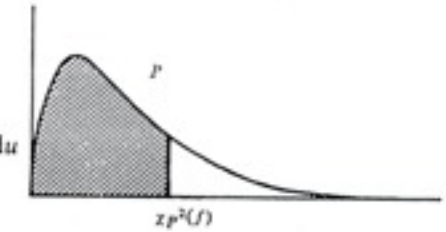


χ^2 \ f	1	2	3	4	5	6	7	8	9	10
0.1	0.248	0.045	0.008	0.001	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.345	0.095	0.022	0.005	0.001	0.000	0.000	0.000	0.000	0.000
0.3	0.416	0.139	0.040	0.010	0.002	0.001	0.000	0.000	0.000	0.000
0.4	0.473	0.181	0.060	0.018	0.005	0.001	0.000	0.000	0.000	0.000
0.5	0.520	0.221	0.081	0.026	0.008	0.002	0.001	0.000	0.000	0.000
0.6	0.561	0.259	0.104	0.037	0.012	0.004	0.001	0.000	0.000	0.000
0.7	0.597	0.295	0.127	0.049	0.017	0.006	0.002	0.000	0.000	0.000
0.8	0.629	0.330	0.151	0.062	0.023	0.008	0.003	0.001	0.000	0.000
0.9	0.657	0.362	0.175	0.075	0.030	0.011	0.004	0.001	0.000	0.000
1.0	0.683	0.393	0.199	0.090	0.037	0.014	0.005	0.002	0.001	0.000
1.1	0.706	0.423	0.223	0.106	0.046	0.01[illegible]	0.007	0.002	0.001	0.000
1.2	0.727	0.451	0.247	0.122	0.055	0.023	0.009	0.003	0.001	0.000
1.3	0.746	0.478	0.271	0.139	0.065	0.0[illegible]	0.012	0.004	0.002	0.001
1.4	0.763	0.503	0.294	0.156	0.076	0.034	0.014	0.006	0.002	0.001
1.5	0.779	0.528	0.318	0.173	0.087	0.041	0.018	0.007	0.003	0.00[illegible]
1.6	0.794	0.551	0.341	0.191	0.099	0.047	0.021	0.009	0.004	0.001
1.7	0.808	0.573	0.363	0.209	0.111	0.055	0.025	0.011	0.005	0.002
1.8	0.820	0.593	0.385	0.228	0.124	0.063	0.030	0.013	0.006	0.002
1.9	0.832	0.613	0.407	0.246	0.137	0.071	0.035	0.016	0.007	0.003
2.0	0.843	0.632	0.428	0.264	0.151	0.080	0.040	0.019	0.009	0.004
3.0	0.917	0.777	0.608	0.442	0.300	0.191	0.115	0.066	0.036	0.019
4.0	0.954	0.865	0.739	0.594	0.451	0.323	0.220	0.143	0.089	0.053
5.0	0.975	0.918	0.828	0.713	0.584	0.456	0.340	0.242	0.166	0.109
6.0	0.986	0.950	0.888	0.801	0.694	0.577	0.460	0.353	0.260	0.185
7.0	0.992	0.970	0.928	0.864	0.779	0.679	0.571	0.463	0.363	0.275
8.0	0.995	0.982	0.954	0.908	0.844	0.762	0.667	0.567	0.466	0.371
9.0	0.997	0.989	0.971	0.939	0.891	0.826	0.747	0.658	0.563	0.468
10.0	0.998	0.993	0.981	0.960	0.925	0.875	0.811	0.735	0.650	0.560
11.0	0.999	0.996	0.988	0.973	0.949	0.912	0.861	0.798	0.7[illegible]4	0.642
12.0	0.999	0.998	0.993	0.983	0.965	0.938	0.899	0.849	0.7[illegible]7	0.715
13.0	1.000	0.998	0.995	0.989	0.977	0.957	0.928	0.888	0.837	0.776
14.0	1.000	0.999	0.997	0.993	0.984	0.970	0.949	0.918	0.878	0.827
15.0	1.000	0.999	0.998	0.995	0.990	0.980	0.964	0.941	0.909	0.868
16.0	1.000	1.000	0.999	0.997	0.993	0.986	0.975	0.958	0.933	0.900
17.0	1.000	1.000	0.999	0.998	0.995	0.991	0.983	0.970	0.951	0.926
18.0	1.000	1.000	1.000	0.999	0.997	0.994	0.988	0.979	0.965	0.945
19.0	1.000	1.000	1.000	0.999	0.998	0.996	0.992	0.985	0.975	0.960
20.0	1.000	1.000	1.000	1.000	0.999	0.997	0.994	0.990	0.982	0.971

Tafel F – 5
Quantile der χ^2–Verteilung.

Tabelliert sind die Werte $\chi^2_P(f)$, definiert durch

$$P = \frac{1}{\Gamma(\frac{1}{2}f)\,2^{\frac{1}{2}f}} \int_0^{\chi_P^2(f)} u^{-\frac{1}{2}f-1}\, e^{-\frac{1}{2}u}\, du$$



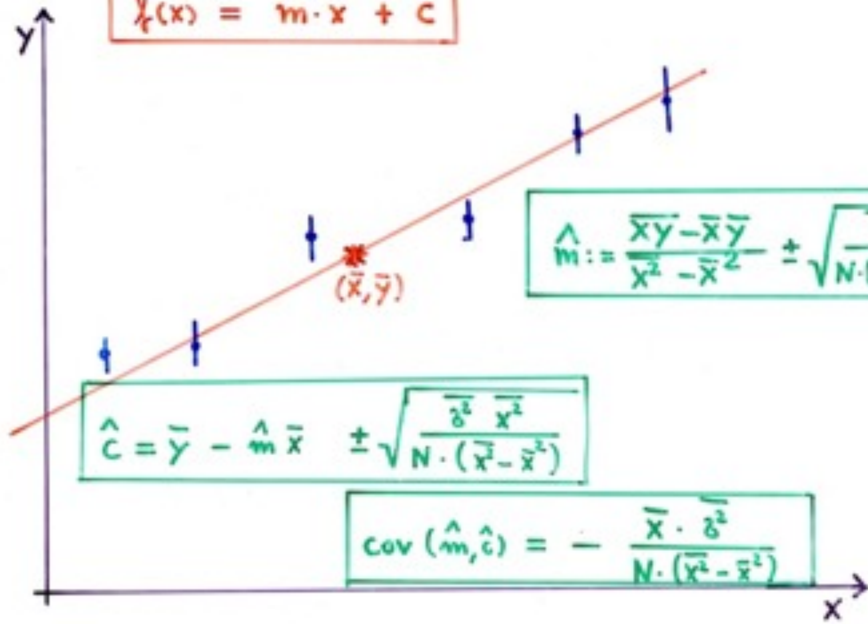
	P								
f	0.100	0.500	0.700	0.8C0	0.900	0.950	0.990	0.995	0.999
1	0.02	0.45	1.07	1.64	2.71	3.84	6.63	7.88	10.83
2	0.21	1.39	2.41	3.22	4.61	5.99	9.21	10.60	13.82
3	C.58	2.37	3.66	4.64	6.25	7.81	11.34	12.84	16.27
4	1.06	3.36	4.88	5.99	7.78	9.49	13.28	14.86	18.47
5	1.61	4.35	6.06	7.29	9.24	11.07	15.09	16.75	20.51
6	2.20	5.35	7.23	8.56	10.64	12.59	16.81	18.55	22.46
7	2.83	6.35	8.38	9.80	12.02	14.07	18.48	20.28	24.32
8	3.49	7.34	9.52	11.03	13.36	15.51	20.09	21.95	26.12
9	4.17	8.34	10.66	12.24	14.68	16.92	21.67	23.59	27.88
10	4.87	9.34	11.78	13.44	15.99	18.31	23.21	25.19	29.59
11	5.58	10.34	12.90	14.63	17.27	19.68	24.72	26.76	31.26
12	6.30	11.34	14.C1	15.81	18.55	21.03	26.22	28.30	32.91
13	7.04	12.34	15.12	16.98	19.81	22.36	27.69	29.82	34.53
14	7.79	13.34	16.22	18.15	21.06	23.68	29.14	31.32	36.12
15	8.55	14.34	17.32	19.31	22.31	25.00	30.58	32.80	37.70
16	9.31	15.34	18.42	20.47	23.54	26.30	32.00	34.27	39.25
17	1C.09	16.34	19.51	21.61	24.77	27.59	33.41	35.72	40.79
18	10.86	17.34	2C.60	22.76	25.99	28.87	34.81	37.16	42.31
19	11.65	18.34	21.69	23.90	27.20	30.14	36.19	38.58	43.82
20	12.44	19.34	22.77	25.04	28.41	31.41	37.57	40.00	45.31
21	13.24	20.34	23.86	26.17	29.61	32.67	38.93	41.40	46.80
22	14.C4	21.34	24.94	27.30	3C.81	33.92	40.29	42.80	48.27
23	14.85	22.34	26.02	28.43	32.01	35.17	41.64	44.18	49.73
24	15.66	23.34	27.10	29.55	33.20	36.42	42.98	45.56	51.18
25	16.47	24.34	28.17	30.68	34.38	37.65	44.31	46.93	52.62
26	17.29	25.34	29.25	31.79	35.56	38.89	45.64	48.29	54.05
27	18.11	26.34	30.32	32.91	36.74	40.11	46.96	49.64	55.48
28	18.94	27.34	31.39	34.C3	37.92	41.34	48.28	50.99	56.89
29	19.77	28.34	32.46	35.14	39.09	42.56	49.59	52.34	58.30
30	20.60	29.34	33.53	36.25	40.26	43.77	50.89	53.67	59.70
40	29.05	39.33	44.17	47.27	51.80	55.76	63.70	66.76	73.39
50	37.69	49.33	54.72	58.16	63.17	67.51	76.16	79.49	86.66
60	46.46	59.33	65.23	68.97	74.40	79.08	88.38	91.95	9[illegible].61
70	55.33	69.33	75.69	79.71	85.53	90.53	100.43	104.21	112.32
80	64.28	79.33	86.12	9C.40	96.58	101.88	112.33	116.32	124.84
90	73.29	89.33	96.52	101.05	107.56	113.15	124.12	128.30	137.21
100	82.36	95.33	106.91	111.67	118.50	124.34	135.81	140.17	149.45

Zusammenfassung: Ausgleichsgerade
Methode der kleinsten Quadrate

geg.: Wertepaare $x_i \qquad i = 1,2,\dots,N$

$y_i \pm \sigma_i \quad i = 1,2,\dots,N$

$\sigma_i \mathrel{\hat{=}}$ Meßfehler



$$\boxed{f(x) = m \cdot x + c}$$

$$\boxed{\hat{m} := \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - \bar{x}^2} \pm \sqrt{\frac{\overline{\sigma^2}}{N \cdot (\overline{x^2} - \bar{x}^2)}}}$$

$$\boxed{\hat{c} = \bar{y} - \hat{m}\,\bar{x} \pm \sqrt{\frac{\overline{\sigma^2}\;\overline{x^2}}{N \cdot (\overline{x^2} - \bar{x}^2)}}}$$

$$\boxed{\mathrm{cov}(\hat{m},\hat{c}) = -\frac{\bar{x} \cdot \overline{\sigma^2}}{N \cdot (\overline{x^2} - \bar{x}^2)}}$$

Wobei:

$$\bar{x} = \frac{1}{N} \cdot \sum x_i$$

$$\bar{y} = \frac{\sum y_i / \sigma_i^2}{\sum 1/\sigma_i^2}$$

$$\overline{\sigma^2} = \frac{N}{\sum 1/\sigma_i^2}$$

χ^2 der besten Anpassung:

$$\boxed{\chi^2_{min} = \frac{(\overline{y^2} - \bar{y}^2)}{\overline{\sigma^2}} \left(1 - \rho^2(x,y)\right)}$$

Beispiel: Ausgleichsgerade, Matrixschreibweise

$$y(x) = m \cdot x + c = a_1 + a_2 \cdot x \qquad \vec{\lambda} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

n Meßpunkte
2 Parameter

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

Annahme:
gleiche Fehler σ für alle Meßpunkte,
keine Korrelationen

$$\Rightarrow \quad C = \begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & \\ \vdots & & \ddots & \\ & & & \sigma^2 \end{pmatrix}$$

Lösung: $\vec{\lambda} = (A^t C^{-1} A)^{-1} A^t C^{-1} \cdot \vec{y}$

$$C = \sigma^2 \cdot \mathbb{1} \Rightarrow \quad \vec{\lambda} = \sigma^2 (A^t A)^{-1} \frac{1}{\sigma^2} A^t \cdot \vec{y}$$

$$\Leftrightarrow \quad \vec{\lambda} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \sum 1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1}}_{=\frac{1}{n(\overline{x^2}-\bar{x}^2)} \begin{pmatrix} \overline{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$= \frac{1}{n(\overline{x^2}-\bar{x}^2)} \begin{pmatrix} \overline{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\Rightarrow \boxed{a_2 = \hat{m} = \frac{1}{n \cdot (\overline{x^2}-\bar{x}^2)} \left(-\bar{x} \cdot \sum y_i + 1 \cdot \sum x_i y_i\right) = \frac{-\bar{x}\bar{y} + \overline{xy}}{(\overline{x^2}-\bar{x}^2)}}$$

Fehlermatrix der Parameter $\vec{\lambda}$: $\quad C_\lambda = (A^t C^{-1} A)^{-1}$

$$= \sigma^2 (A^t A)^{-1} = \frac{\sigma^2}{n(\overline{x^2}-\bar{x}^2)} \begin{pmatrix} \overline{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$$

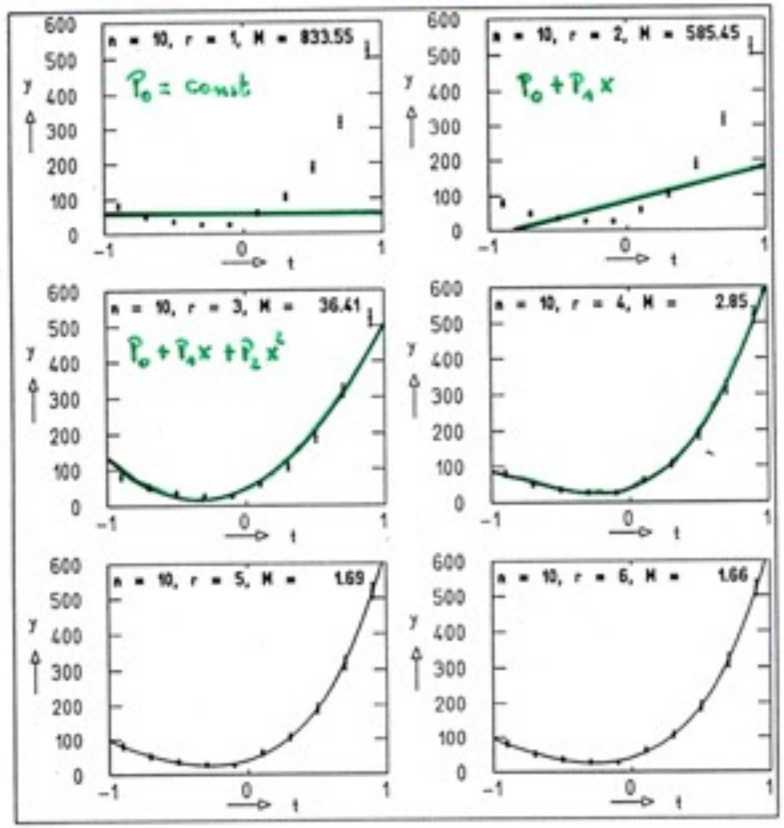
Beispiel: $\boxed{\text{Polynom - Anpassung}}$

$$y(x) = a_1 + a_2 \cdot x + a_2 \cdot x^2$$

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix}$$

$$\vec{\lambda} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \sum 1 & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

Beispiel: Polynomanpassung an Meßdaten



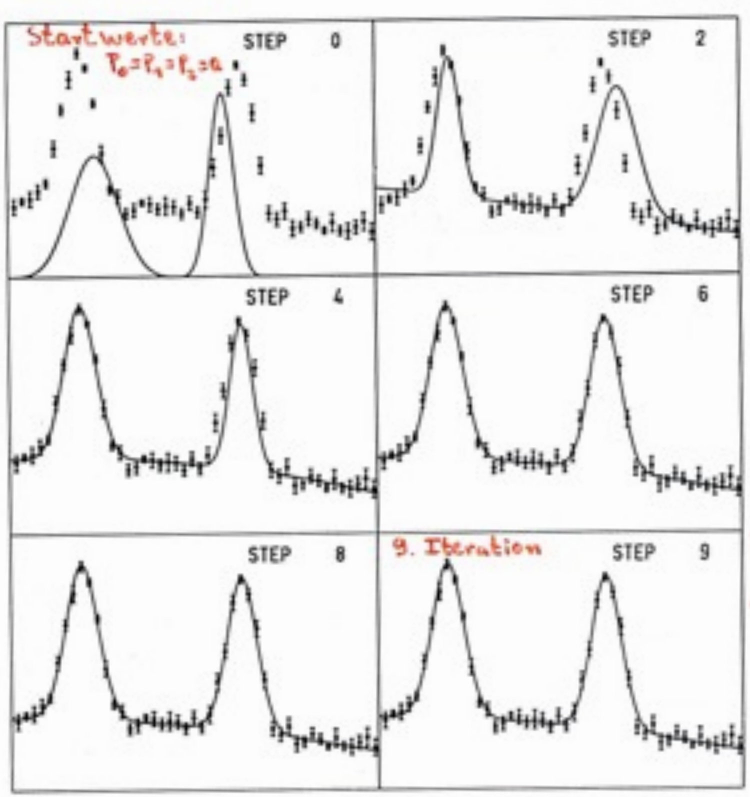
Fit-Parameter:

r	P_0	P_1	P_2	P_3	P_4	P_5	f	χ^2
1	57.85						9	833.55
2	82.66	99.10					8	585.45
3	47.27	185.96	273.61				7	36.41
4	37.94	126.55	312.02	137.59			6	2.85
5	39.62	119.10	276.49	151.91	52.60		5	1.68
6	39.88	121.39	273.19	136.58	56.90	16.72	4	1.66

Parameter

Freiheitsgrade

komplizierterer, nicht-linearer Fall.



Abbruchbed. $\Delta\chi^2 < \Delta_{min}$

Bild 9.9: Schrittweise Annäherung der anzupassenden Funktion an die Meßwerte.

Zwei Signale : Gauß-Form
Untergrund : unbekannt, stetig → Polynom-Ansatz

$$f(x,\vec{\lambda}) = P_0 + P_1 \cdot x + P_2 \cdot x^2 + P_3 \cdot \exp\{-(P_4 - x)^2 / 2 P_5^2\} + P_6 \cdot \exp\{-(P_7 - x)^2 / 2 \cdot P_8^2\}$$

$$\vec{\lambda} = \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_8 \end{pmatrix}$$

9 Parameter

→ benutze zuverlässige, erprobte Computer-Programme zur Minimum-Bestimmung.

z.B. $\boxed{\text{MINUIT}}$